

Differentiating an integral (Leibniz's formula)

Here we use the result that $\frac{d}{dx} \int_a^b f(x, t) dt = \int_a^b \frac{\partial f}{\partial x}(x, t) dt$, where a and b are constants.

This can be thought of as taking the $\frac{d}{dx}$ inside the integral. This can be generalised to:

$$\frac{d}{dx} \int_{a(x)}^{b(x)} f(x, t) dt = b'(x)f(x, b(x)) - a'(x)f(x, a(x)) + \int_{a(x)}^{b(x)} \frac{\partial f}{\partial x}(x, t) dt$$

This formula can be found in the *Tables*.

The formula is very useful in cases where the integral cannot easily be evaluated

directly, for example $\frac{d}{dx} \int_0^{\infty} e^{-xt^2} dt$.

Example

Evaluate $\frac{d}{dx} \int_0^x x^2 + t \, dt$.

Solution

Here $a(x)=0$, $b(x)=x$, $f(x,t)=x^2+t$, so:

$$\begin{aligned} \frac{d}{dx} \int_0^x x^2 + t \, dt &= 1(x^2 + x) - 0 + \int_0^x 2x \, dt \\ &= x^2 + x + 2x^2 \\ &= 3x^2 + x \end{aligned}$$

In this case we can show that this is the same as integrating directly:

$$\int_0^x x^2 + t \, dt = \left[x^2 t + \frac{1}{2} t^2 \right]_0^x = x^3 + \frac{1}{2} x^2$$

$$\text{So } \frac{d}{dx} \int_0^x x^2 + t \, dt = \frac{d}{dx} \left(x^3 + \frac{1}{2} x^2 \right) = 3x^2 + x.$$

Question 1.1

Evaluate $\frac{d}{dx} \int_0^{2x+3} [(x+1)^2 + tx] \, dt$.

Solutions

Solution 1.1

We need to find $\frac{d}{dx} \int_0^{2x+3} (x+1)^2 + tx \, dt$, so comparing with the general equation:

$$a(x) = 0, \quad b(x) = 2x + 3, \quad \text{and} \quad f(x, t) = (x+1)^2 + tx$$

giving:

$$a'(x) = 0, \quad b'(x) = 2, \quad \text{and} \quad \frac{\partial f}{\partial x} = 2(x+1) + t$$

So:

$$\begin{aligned} \frac{d}{dx} \int_0^{2x+3} (x+1)^2 + tx \, dt &= 2 \left[(x+1)^2 + x(2x+3) \right] - 0 + \int_0^{2x+3} 2(x+1) + t \, dt \\ &= 2(3x^2 + 5x + 1) + \left[2(x+1)t + \frac{1}{2}t^2 \right]_0^{2x+3} \\ &= 2(3x^2 + 5x + 1) + \left(2(x+1)(2x+3) + \frac{1}{2}(2x+3)^2 \right) \\ &= 12x^2 + 26x + 12\frac{1}{2} \end{aligned}$$